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ELEMENTARY TREATMENT OF THE PROBLEM OF TWO BODIES.

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The deduction of the motion of the planets, in accordance with the laws of Kepler, from the principle of universal gravitation, is important, not only on account of the extensive role this theory plays in Astronomy, but also for its interest, in a historical point of view, as Newton's principal discovery. Hence it is desirable that the demonstration be made as elementary and as brief as possible, in order that it may be brought within the comprehension of the largest number of persons.

The polar equation of the conic section, referred to a focus as pole,

$$r = \frac{a(1 - e^2)}{1 + e\cos(\lambda - \omega)},$$

is well known; a denotes half the greater axis, e the eccentricity and ω the angle made by the axis with the line from which λ is measured. It will be advantageous to replace $a(1 - e^2)$ by p, p being the parameter, also to put

$$\alpha = e\cos\omega, \qquad \beta = e\sin\omega.$$

Thus the equation becomes

$$r + \alpha r\cos\lambda + \beta r\sin\lambda = p.$$

Hence it is plain that the equation, in terms of rectangular coordinates, the origin being at a focus, but the axes of coordinates having any direction we please, is

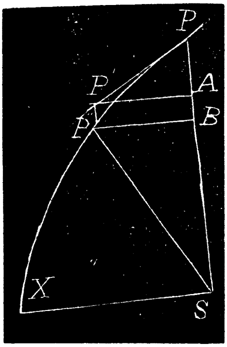
$$\sqrt{x^2 + y^2} + \alpha x + \beta y = p \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

We will take it for granted that the reader is acquainted with the following theorems, since they are demonstrated in all books on mechanics, even the most elementary:

In determining the relative motion of one body about another, it suffices to regard the latter as fixed, and to attribute to it a mass equal to the sum of the masses, and then to suppose the moving body without mass.

When a body describes a plane curve, and the radius vector, drawn from a fixed point in the plane of the curve, passes over equal areas in equal times, (which we shall express by saying that the areolar velocity about the fixed point is constant), the force acts always in the direction of the radius: and the converse.

Let now a body describe a conic section about another occupying a focus, the areolar velocity about this focus being constant: it is required to determine the force acting.



In the figure, let PP″X be an arc of the conic section so described, S being the focus. Let P and P″ be any two points on the curve at an indeterminate but small distance from each other. Draw SP, PP′ a tangent at P, P″P′ parallel to, and P′A and P″B perpendicular to SP. Let SP be taken as the axis of y, and SX perpendicular to it, as the axis of x. The coordinates of P are then, $x = 0$, $y = \mathrm{SP} = r$; substituting these in the equation of the cure, we get

$$(1 + \beta)r = p \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Since the coordinate x is here supposed always very small, the term $\sqrt{x^2 + y^2}$ in (1) can be expanded, by the binomial theorem, in a series of ascending powers of x. Neglecting x^4 and higher powers we get

$$y + \frac{x^2}{2y} + ax + \beta y = p,$$

or as y differs from r only by a quantity of the order of x, by neglecting x^3,

$$y + \frac{x^2}{2r} + ax + \beta y = p, \qquad y = \frac{p - ax - \frac{x^2}{2r}}{1 + \beta}.$$

Or by (2)

$$y = r - \frac{\alpha}{1 + \beta} x - \frac{x^2}{2p}.$$

This is the value of y from (1), expanded in a series of ascending powers of x, the cubes and higher powers being omitted. The equation

$$y = r - \frac{\alpha}{1 + \beta} x$$

belongs to a right line, which can be nothing else than the tangent PP′. Hence it is plain, from the figure, that taking P″B = P′A = x,

$$\text{tan. PP'A} = \frac{\alpha}{1 + \beta} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

$$\text{PA} = \frac{\alpha}{1 + \beta} x,$$

$$\text{P'P''} = \text{AB} = \frac{x^2}{2p} \quad \ldots\ldots\ldots\ldots\ldots\ldots (4),$$

the last equation being only approximate, but more and more nearly true as P″B or x becomes smaller.

Let F denote the force acting on the moving body, and t the small interval of time in which it passes from P to P″. Then we have

$$\text{P'P''} = \frac{x^2}{2p} = \tfrac{1}{2}\text{F}t^2.$$

If we denote double the areolar velocity by h, since P″B = x is very small, we have

$$\text{SP.P''B} = rx = ht.$$

Eliminating t from these equations, we get

$$\text{F} = \frac{h^2}{pr^2}.$$

Since there is no limit to the supposed smallness of x and t, this equation is rigorously exact. The force is then inversely as the square of the radius-vector, and its intensity at the unit of distance is found simply by dividing the square of double the areolar velocity by the semi-parameter. It is evidently attractive, except when, the motion being in a hyperbola, the focus, about which the areolar velocity is constant, is the exterior, in which case it is repulsive.

Taking up the inverse problem, let a body start from P towards P′ with a velocity v which would carry it to the latter point in the time t, and let it be subjected to the action of a force varying inversely as the square of its distance from a second body supposed fixed at S: it is required to fiud the curve described.

Let the masses of the bodies. measured by the velocities they are able to communicate, by their action, in the unit of time and at the unit of distance, be denoted severally by m and M. The force acting at P is then

$$\frac{\mathrm{M} + m}{\mathrm{SP}^2} = \frac{\mathrm{M} + m}{r^2},$$

and, if at the end of the time t, the body is at P″ instead ot P′, we must have

$$\mathrm{P'P''} = \tfrac{1}{2}\frac{\mathrm{M} + m}{r^2}\, t^2.$$

But, as before, the constancy of the areolar velocity gives

$$rx = ht.$$

Whence

$$\mathrm{P'P''} = \frac{\mathrm{M} + m}{2h^2}\, x^2.$$

This equation coincides with (4), if we suppose

$$p = \frac{h^2}{\mathrm{M} + m} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5).$$

Let now a conic section, having this value for its semi-parameter, be described with S as focus and touching PP′ at P. That this is possible, is evident from the general equation (1); here are only two unknowns, α and β, to be determined, and they are given by equations (2) and (3), whence we see the solution is always unique. A body, moving upon this conic section, would have, at the point P, the same velocity, and the same direction of motion, and be subjected to the action of an equa force, and having the same law of variation, as the moving body in the problem. Hence, if the path of the latter is thoroughly determinate, and it would be absurd to suppose otherwise, the conic section just described must be the curve sought.

We can easily find the elements of this conic section. Thus let the angle P′PS be denoted by ψ, then evidently

$$h = rv\sin\psi,$$

which, substituted in (5), gives the value of $p = a(1 - e^2)$; next α and β, which we recall stand for $e\cos\omega$ and $e\sin\omega$, are given by (2) and (3). That is

$$a(1 - e^2) = \frac{r^2v^2\sin^2\psi}{M + m},$$

$$e\cos\omega = \frac{rv^2\sin\psi\cos\psi}{M + m},$$

$$e\sin\omega = \frac{rv^2\sin^2\psi}{M + m} - 1,$$

whence we derive

$$e^2 = 1 - 2\,\frac{rv^2\sin^2\psi}{M + m} + \frac{r^2v^4\sin^2\psi}{(M + m)^2}, \qquad \frac{1}{a} = \frac{2}{r} - \frac{v^2}{M + m},$$

consequently the greater axis, and the species of conic section described, are independent of ψ. We have an ellipse, a parabola, or a hyperbola, according as v^2 is less, equal to, or greater than $2\,\dfrac{M + m}{r}$.

From the last equation $v^2 = (M + m)\left(\dfrac{2}{r} - \dfrac{1}{a}\right)$..............(6),

which may evidently be taken as a general expression for the square of the velocity, if r denote the general radius-vector.

Also from (5), $$h = \sqrt{(M + m)p}.$$

Thus, in different orbits, the areolar velocities are as the square-roots of the parameters, and as the square-roots of the sums of the masses. In an elliptic orbit, if T denote the time of revolution, the double of the area of the whole ellipse

$$hT = 2\pi a^2\sqrt{1 - e^2} = 2\pi a^{\frac{3}{2}}\sqrt{p},$$

whence $$T = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{M + m}}.$$

Thus the theorem that, provided the sum of the masses remain the

same, the squares of the periodic times in different orbits are as the cubes of the greater axes.

The mean angular velocity is usually denoted by n, thus,

$$n = \frac{2\pi}{T} = \sqrt{\frac{M + m}{a^3}}.$$

It is customary with astronomers to assume the earth's mean distance from the sun as the linear unit. If M and m are the masses severally of the sun and earth, and m', a' and n' belonging to another planet are introduced, the mean distance of the last is given by the equation

$$a' = \left[\frac{1 + \frac{m'}{M}}{1 + \frac{m}{M}} \frac{n^2}{n'^2}\right]^{\frac{1}{3}}.$$

To complete the subject it is necessary to notice a particular case of the problem, viz., when $\psi = 0$. Here the motion is in a right line, and from (6) it appears the velocity is infinite when the body arrives at S. As the existence of another body here ought not to be considered, at least in a mathematical sense, as an obstacle to its further motion, it is plain the body will pass beyond and move in the same right line until its velocity is reduced to zero, when it will return on its path, which will thus be a portion of a right line of which S is the middle point. This cannot be considered as a degenerate form of a conic section of which S is the focus. For when an ellipse is varied by augmenting the eccentricity but maintaining the greater axis constant, at the point the first has attained the limit unity, the ellipse has degenerated into two equal portions of right lines overlapping each other and having their extremities on one side in the point S. Hence this case must be regarded as a singular solution. However most of the properties of motion can be deduced from those of elliptic motion. Thus, if the length of the whole path be denoted by $4a$, the duration of an oscillation will be

$$\frac{2\pi a^{\frac{3}{2}}}{\sqrt{M + m}}.$$

Whence we gather that the time, in which a planet, at rest at its mean distance, would fall to the sun, is found by dividing its periodic time by $4\sqrt{2}$.